

Achieving Envy-Freeness through Items Sale

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Abstract

We consider a fair division setting of allocating indivisible items to a set of agents. In order to cope with the well-known impossibility results related to the non-existence of envy-free allocations, we allow the option of selling some of the items so as to compensate envious agents with monetary rewards. In fact, this approach is not new in practice, as it is applied in some countries in inheritance or divorce cases. A drawback of this approach is that it may create a value loss, since the market value derived by selling an item can be less than the value perceived by the agents. Therefore, given the market values of all items, a natural goal is to identify which items to sell so as to arrive at an envy-free allocation, while at the same time maximizing the overall social welfare. Our work is focused on the algorithmic study of this problem, and we provide both positive and negative results on its approximability. When the agents have a commonly accepted value for each item, our results show a sharp separation between the cases of two or more agents. In particular, we establish a PTAS for two agents, and we complement this with a hardness result, that for three or more agents, the best approximation guarantee is provided by essentially selling all items. This hardness barrier, however, is relieved when the number of distinct item values is constant, as we provide an efficient algorithm for any number of agents. We also explore the generalization to heterogeneous valuations, where the hardness result continues to hold, and where we provide positive results for certain special cases.

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40 **1 Introduction**

41 *Fair division* refers to the algorithmic question of allocating resources or tasks to a set of
42 agents according to some justice criteria. It is by now a prominent area within Algorithmic
43 Game Theory and Computational Social Choice, [12, Part II], dating back to the origins of
44 the civil society. One of the most natural and well studied notions of fairness is *envy-freeness*
45 [18]: a division is envy-free if everyone thinks that her share is at least as valuable as the
46 share of any other agent. In the presence of indivisible items however, obtaining an envy-free
47 allocation is much more challenging [17], and it is well known that, in the majority of cases,
48 envy-free divisions do not exist.

49 An approach that has been followed by several works, in order to cope with these
50 existential issues, is to focus on relaxations of envy-freeness (for more on this we refer to
51 our related work section). Another natural direction that comes into mind is to insist on
52 envy-freeness but provide some compensation (e.g., monetary) to the agents who may feel
53 unhappy by a proposed division. Such models have been considered in the literature, where
54 money is either coming as an external subsidy from a third party or is already part of the
55 initial endowment. Under this setting, [20] investigated the question of determining the
56 minimum amount of money needed to obtain an envy-free division.

57 In this work, we also allow for monetary rewards, but we choose a different approach,
58 as already initiated in [22]: we require that the money used to compensate the envious
59 agents has to be raised from the set of available items, by selling some of them. This is what
60 happens, for instance, in inheritance division. To provide some examples, as stated in Article
61 n.9 of the New York Laws - Real Property Actions and Article n.720 of the Italian Civil
62 Code, whenever an agreement is not possible, part of the inheritance can be sold through
63 an auction. The same practice is also used in divorce settlements. Clearly, envy-freeness
64 is then always feasible by selling, if needed, the whole inheritance, and equally sharing the
65 proceeds. However, the amount of money raised by this process can be fairly below the
66 real value of the sold items for at least two reasons. First, the bidders who participate in
67 this type of auctions usually aim at winning items at very low prices; secondly, running an
68 auction bears organizational costs which need to be subtracted from the proceeds. Thus, it
69 is in the interest of the heirs to determine an envy-free division by selling assets with as little
70 value loss as possible. This gives rise to an interesting optimization problem of determining
71 which items to sell so as to arrive at an envy-free allocation, with optimal social welfare.
72 Algorithmically, this question has been largely unexplored, with the exception of a particular
73 case handled in [22].

74 **1.1 Contribution**

75 Assuming that we are given the *market value* of each item as input, i.e., the money that can
76 be raised by selling it, we embark on a thorough investigation of algorithmic and complexity
77 questions for our problem and provide an almost tight set of results.

78 We start in Section 3 with the case where all agents have the same value for each item.
79 After establishing NP-hardness, which can be easily shown even for 2 agents, our main results
80 exhibit a sharp separation on the approximability between the cases of $n = 2$ and $n \geq 3$
81 agents. In particular, we prove that, with at least three agents, no polynomial time algorithm
82 can obtain a solution that performs better than the one which sells all items, unless $P = NP$.
83 On the other hand, for two agents, we are able to design a polynomial time approximation
84 scheme (PTAS), under the assumption that the market value of each item is not smaller
85 than half of the common agents' value. The idea behind the PTAS is to enumerate all partial

86 allocations of the most valuable items, whose number is a constant depending on the desired
87 approximation guarantee. Each such partial allocation, which consists of the two bundles
88 assigned to the agents together with the bundle of sold items, is then completed processing
89 the remaining items by non-increasing value. At every step, the next item is allocated to
90 the agent having the lower valued bundle, until we reach a situation where it is possible to
91 equalize the two bundles by using the money raised from the already sold items and from
92 selling a subset of the not-yet-processed ones. The main technical effort is needed to show
93 that, if this condition occurs, then the final allocation can be made envy-free at the expense
94 of a negligible loss of social welfare, while, if the condition never occurs, then it is not possible
95 to obtain an envy-free solution from the starting partial allocation. Finally, our last result in
96 Section 3 is the design of a dynamic programming algorithm which runs in polynomial time
97 when the number of distinct item values is constant; this assumption is in line with several
98 other recent works on fair-division, e.g., [5, 1].

99 In Section 4, we then move to the case where agents can have heterogeneous valuations.
100 While all computational barriers from Section 3 carry over to this case as well, we are able
101 to obtain two additional positive results. First, we focus on the setting where the values that
102 an agent i has for the items lie in an interval of the form $[x_i, \beta x_i]$, where β is common across
103 all agents. This means, essentially, that each agent attributes the same value to all items,
104 up to a factor of β . For a constant number of agents, and for a constant value of β , we are
105 able to design again a PTAS. This is very different from the PTAS of Section 3 and is based
106 on an appropriate combination of two main ideas. First, by using a linear programming
107 formulation, we compute a fractional solution with a bounded number of fractionally assigned
108 items. Then, we apply a "reverse" version of the envy cycle elimination algorithm [24], so as
109 to decide which items to sell, in addition to the fractional ones. We believe that this could be
110 of independent interest for other allocation problems as well. Finally, at the end of Section 4,
111 where we drop the assumption on β being constant, we also provide a pseudo-polynomial
112 time algorithm.

113 Due to lack of space, all omitted proofs are deferred to the full version of this work.

114 1.2 Related Work

115 In terms of the model that we study, the work most related to ours is [22]. Their main focus
116 however is not algorithmic but instead aims at comparing the *Price of Fairness* with and
117 without selling items (defined as the ratio of the optimal social welfare versus the welfare
118 attainable at an envy-free allocation). Their work also includes one algorithmic result,
119 namely a PTAS, but for a case of only two heterogeneous agents, and under assumptions
120 that are incomparable to our results of Section 4, where we consider any constant number of
121 heterogeneous agents.

122 The use of monetary compensations, as a means to achieve envy-freeness with indivisible
123 goods, has also been studied from various other angles in the literature, dating back e.g.,
124 to [27, 25]. The models that have been studied most often consider cases where each agent
125 receives at most a single good (motivated by rent division) in addition to money. Given an a
126 priori fixed amount of money, [2] yields an algorithm for determining an envy-free allocation.
127 Improved algorithms were also provided in follow up works in [6, 23], and a more general
128 model was considered in [9]. More recently, [20] take an optimization approach of minimizing
129 the amount of required money for achieving envy-freeness (and without any restrictions in
130 the items given to an agent). The main conceptual difference with our work is that in all
131 these models, money is viewed as an already existing subsidy, whereas in our case it comes
132 from selling some of the available items, which also leads to welfare loss.

133 Moving away from monetary rewards, there have been by now various other approaches
 134 for addressing the lack of envy-free allocations under indivisible items. The most popular one
 135 is the quest for relaxed notions of fairness, tailored for indivisible items. The notions of envy
 136 freeness up to 1 good (EF1) [13, 24], and envy freeness up to any good (EFX) [16, 19] are the
 137 two most representative examples, that have motivated a vast amount of recent works. An
 138 alternative direction was initiated in [14] where allocations satisfying the EFX criterion (and
 139 with high Nash welfare) were shown to exist when some items remain unallocated; giving
 140 birth to fairness *with charity*. For an overview of these notions and the relevant results, we
 141 refer the reader to the recent survey [3] and the references therein.

142 Finally, another relevant line of works concerns the quantification of welfare performance
 143 subject to fairness constraints. One way to formalize this is via the Price of Fairness [15],
 144 defined in the beginning of this section. The Price of Fairness w.r.t. other fairness criteria,
 145 such as relaxations of envy-freeness was later studied in [8], and further results with tight
 146 bounds were also given in [7, 21]. We refer again to the survey [3] for a more complete
 147 overview.

148 2 Definitions

149 We consider a set $[m] := \{1, \dots, m\}$ of m indivisible items to be allocated to a set $[n]$ of n
 150 agents. We assume that for every item j , there is a commonly accepted value $v(j)$, by all
 151 agents¹. The vector $\mathbf{v} = (v(1), \dots, v(m))$ induces an additive valuation function $v : 2^{[m]} \mapsto \mathbb{N}$,
 152 so that for every subset $S \subseteq [m]$, the value of S is $v(S) = \sum_{j \in S} v(j)$.

153 An additional choice, instead of allocating all items to the agents, is to sell some of them
 154 in exchange of money. The rationale here is that if allocating all items cannot result in a
 155 fair allocation, we could use monetary compensations from sold items to achieve a more
 156 acceptable outcome. This may come at some value loss, since selling an item in the market
 157 can lead to a lower price than the value perceived by the agents. In particular, we assume
 158 that we are given a *market value* vector $\mathbf{v}_0 = (v_0(1), \dots, v_0(m))$, so that $v_0(j)$ is the monetary
 159 amount that can be obtained by selling item j , with $v_0(j) \leq v(j)$, for every $j \in [m]$. Viewing
 160 the vector \mathbf{v}_0 as inducing an (alternative) additive valuation function, we have that for every
 161 $S \subseteq [m]$, the money obtained by selling the items of S is equal to $v_0(S) = \sum_{j \in S} v_0(j)$.

162 Given an instance defined by a tuple $(n, m, \mathbf{v}, \mathbf{v}_0)$, an *allocation with items sale* is a
 163 partition of $[m]$ into $n + 1$ subsets $\mathbf{X} = (X_0, X_1, \dots, X_n)$, such that, for each $i \in [n]$, X_i
 164 is the bundle allocated to agent i and X_0 is the set of items which are sold. Hence, the
 165 money made from \mathbf{X} is $v_0(X_0)$. The *social welfare* of an allocation with items sale is given
 166 by $SW(\mathbf{X}) = v_0(X_0) + \sum_{i \in [n]} v(X_i)$. We will say that an allocation $\mathbf{X} = (X_0, X_1, \dots, X_n)$
 167 is an *envy-free allocation with items sale* (from now on, simply EF-IS), if there exists a split
 168 of the money $v_0(X_0)$ into n amounts μ_1, \dots, μ_n , such that, for any two agents $i, i' \in [n]$,
 169 $v(X_i) + \mu_i \geq v(X_{i'}) + \mu_{i'}$. Since this needs to hold for any pair of agents, we can simplify
 170 the definition of EF-IS as follows. Define the maximum envy of an agent i under an
 171 allocation \mathbf{X} as $e_i^{max}(\mathbf{X}) = \max_{i' \in [n]} \{v(X_{i'})\} - v(X_i)$ (note that, as i' can be also equal to
 172 i , $e_i^{max}(\mathbf{X}) \geq 0$). Then, an allocation is EF-IS if and only if

$$173 \quad v_0(X_0) \geq \sum_{i \in [n]} e_i^{max}(\mathbf{X}). \quad (1)$$

¹ We start with this modeling choice, as it is common in inheritance or divorce settlements, that items such as land properties or cars have a common value to the agents. In Section 4, we explore extensions beyond this assumption.

174 Hence, whenever the above equation holds, it means that there is enough money to compensate
175 all agents having non-zero maximum envy.

176 We observe that an EF-IS allocation always exists: simply sell all items and share equally
177 all the money. We call this allocation, the *basic* EF-IS allocation. Therefore, this gives rise
178 to the natural optimization problem of finding the best EF-IS allocation in terms of social
179 welfare. This constitutes the focus of our work, and we define it formally below.

180 **BEST-EF-IS:** Given an instance $(n, m, \mathbf{v}, \mathbf{v}_0)$ on n agents, m items, value vector \mathbf{v} and
181 market value vector \mathbf{v}_0 , find an allocation $\mathbf{X} = (X_0, X_1, \dots, X_n)$ that is EF-IS and attains
182 maximum social welfare.

183 3 Hardness and Approximability

184 We begin with defining a parameter that plays a fundamental role in the majority of our
185 results. Given an instance $(n, m, \mathbf{v}, \mathbf{v}_0)$, let $\alpha := \min_{j \in [m]: v(j) > 0} \left\{ \frac{v_0(j)}{v(j)} \right\} \in [0, 1]$, be the
186 largest discrepancy between the market value and the commonly accepted value of any item.
187 We observe the following.

188 ▶ **Observation 1.** *The basic EF-IS allocation is an α -approximation of BEST-EF-IS.*

189 3.1 Hardness Results

190 We address first two extreme cases of the problem. By Observation 1, BEST-EF-IS is trivial
191 when $\alpha = 1$. On the opposite side, when $\alpha = 0$, the problem cannot be approximated up to
192 any finite factor, even for $n = 2$.

193 ▶ **Theorem 2.** *For $n = 2$ and $\alpha = 0$, BEST-EF-IS cannot be approximated up to any finite
194 factor, unless $P = NP$.*

195 For the more interesting cases of $\alpha \in (0, 1)$, a direct reduction from PARTITION yields
196 NP-hardness even for $n = 2$.

197 ▶ **Theorem 3.** *For $n = 2$ and $\alpha \in (0, 1)$, BEST-EF-IS is NP-hard.*

198 The next theorem is our main result from this subsection. It implies that the α -
199 approximate solution achieved by the basic EF-IS allocation is the best we can hope
200 for, when we have at least 3 agents.

201 ▶ **Theorem 4.** *For any $n \geq 3$ and $\alpha \in (0, 1)$, BEST-EF-IS cannot be approximated with a
202 ratio better than $\alpha + \epsilon$, for any constant $\epsilon > 0$, unless $P = NP$.*

203 **Proof.** Fix n , α and ϵ and let $s(\alpha)$ be the number of bits needed to encode α . We show the
204 claim by a gap producing reduction from PARTITION. Consider an instance of PARTITION
205 I made up of $p > \max \left\{ n, s(\alpha), \frac{n}{(n-2)(1-\alpha)}, \frac{2(1-\alpha)}{\epsilon n} \right\}$ positive integers w_1, \dots, w_p , such that
206 $\sum_{j \in [p]} w_j = 2B$. The PARTITION problem asks to find a subset $A \subseteq [p]$ such that $\sum_{j \in A} w_j =$
207 B . Create an instance I' of BEST-EF-IS with $m = p + n$ items such that $v(j) = w_j$ for
208 each $j \in [p]$, $v(p+1) = v(p+2) = (p-1)B$, and $v(j) = pB$ for each $j > p+2$. We call any
209 item of value pB a *big item*, any item of value $(p-1)B$ an *almost big item* and the remaining
210 items *small items*. Note that there are $n-2$ big items and that $\sum_{j \in [m]} v(j) = npB$. The
211 vector \mathbf{v}_0 is defined in such a way that $v_0(j) = \alpha v(j)$ for each $j \in [m]$. Note that, by the
212 choice of p , the representation of I' is polynomial in that of I , for any value of n , α and ϵ .
213 The remaining proof is then completed by establishing the following lemma:

214 ► **Lemma 5.** *If I admits a partition, we can construct in polynomial time an EF-IS allocation*
 215 *\mathbf{X} , with $SW(\mathbf{X}) = npB$. Conversely, if I does not admit a partition, any EF-IS allocation*
 216 *\mathbf{X} satisfies $SW(\mathbf{X}) < (\alpha + \epsilon)npB$.*

217

218 Finally, in the following theorem we show that, if the number of agents is not fixed, the
 219 hardness of approximation holds even if the item values are polynomially bounded in m and
 220 n .

221 ► **Theorem 6.** *For any $\alpha \in (0, 1)$ and $\epsilon > 0$, approximating BEST-EF-IS to better than*
 222 *$\alpha + \epsilon$ is strongly NP-hard.*

223 3.2 A PTAS for Two Agents when $\alpha \geq 1/2$

224 In light of the hardness results shown in the previous subsection, non-trivial approximation
 225 algorithms, without any further assumptions, are possible only for the case of $n = 2$ and
 226 $\alpha \in (0, 1)$. This is the focus of this subsection, and our main result is the design of a
 227 polynomial time approximation scheme (PTAS), under the mild assumption that $\alpha \geq 1/2$
 228 (the market value is never less than 50% of the real value for any item). We feel that this
 229 assumption is not far from what we would expect in practice. If the items (e.g. in an
 230 inheritance or divorce case) had a way too low market value, it would not even make sense
 231 to sell them at all.

232 Within this subsection, we use $(m, \mathbf{v}, \mathbf{v}_0)$ to describe an instance, since $n = 2$. Note that
 233 the condition described in Equation (1) for an allocation $\mathbf{X} = (X_0, X_1, X_2)$ to be EF-IS
 234 simplifies to $v_0(X_0) \geq |v(X_1) - v(X_2)|$. Before illustrating our PTAS, we start with a simple
 235 procedure, called COMPLETE, that will be used as a subroutine of our algorithm. Say that
 236 an instance is *ordered* if items are sorted by non-increasing value. COMPLETE works as
 237 follows: given an ordered instance $I = (m, \mathbf{v}, \mathbf{v}_0)$, an integer $q < m$ and an EF-IS allocation
 238 (X_0, X_1, X_2) restricted to the first q items of I , it sells items $q + 1$ and $q + 2$ (if any), and
 239 then, for each $j \geq q + 3$, item j is assigned to the agent whose bundle, considering only the
 240 items allocated by COMPLETE so far, has the smaller total value, breaking ties in favour of
 241 agent 1.

242 As shown in the lemma below, COMPLETE helps us in extending an EF-IS allocation of
 243 an initial subset of items, to an EF-IS allocation over all the items.

244 ► **Lemma 7.** *Given an ordered instance $I = (m, \mathbf{v}, \mathbf{v}_0)$, an integer $q < m$ and an EF-IS*
 245 *allocation $\mathbf{X} = (X_0, X_1, X_2)$ restricted to the first q items of I , COMPLETE(I, q, \mathbf{X}) returns*
 246 *in $O(m)$ time an EF-IS allocation for I selling items that are worth a total value of at most*
 247 *$v(X_0) + 2v(q + 1)$.*

248 For any $\epsilon > 0$, our algorithm, called SOLVEBESTEF-IS, returns a $(1 - \epsilon)$ -approximation
 249 for BEST-EF-IS. The algorithm relies on an initial brute force enumeration of all possible
 250 allocations with items sale, restricted to the first \bar{q} items, where \bar{q} is a constant that depends
 251 on ϵ . As each item $j \in [\bar{q}]$ can be either assigned to one of the two agents or sold (three
 252 possible choices), there is a total of $3^{\bar{q}}$ possible outcomes. For each outcome, corresponding to
 253 a partial allocation with items sale $\mathbf{X} = (X_0, X_1, X_2)$ restricted to the first \bar{q} items, function
 254 EXTEND is invoked. This function takes an ordered instance $I = (m, \mathbf{v}, \mathbf{v}_0)$, an integer $q < m$
 255 and an allocation $\mathbf{X} = (X_0, X_1, X_2)$ restricted to the first q items of I . Running it with
 256 $q = \bar{q}$, it checks whether \mathbf{X} can be extended to the remaining $m - \bar{q}$ items, that is, *without*
 257 *altering the allocation of the first \bar{q} items*, so as to yield an EF-IS allocation for I . If this

Algorithm 1 COMPLETE($m, \mathbf{v}, \mathbf{v}_0, q, X_0, X_1, X_2$)

Input: an ordered instance $I = (m, \mathbf{v}, \mathbf{v}_0)$, an integer $q < m$ and an EF-IS allocation (X_0, X_1, X_2) restricted to the first q items of I

Output: an EF-IS allocation for I

```

1:  $Y_1 \leftarrow \emptyset, Y_2 \leftarrow \emptyset$ 
2: if  $q = m - 1$  then
3:    $Y_0 \leftarrow \{m\}$ 
4: else
5:    $Y_0 \leftarrow \{q + 1, q + 2\}$ 
6: end if
7: for  $j \leftarrow q + 3$  to  $m$  do
8:   if  $v(Y_1) \leq v(Y_2)$  then
9:      $Y_1 \leftarrow Y_1 \cup \{j\}$ 
10:  else
11:     $Y_2 \leftarrow Y_2 \cup \{j\}$ 
12:  end if
13: end for
14: return  $(X_0 \cup Y_0, X_1 \cup Y_1, X_2 \cup Y_2)$ 

```

Algorithm 2 SOLVEBESTEF-IS($m, \mathbf{v}, \mathbf{v}_0$)

Input: an instance $I = (m, \mathbf{v}, \mathbf{v}_0)$, with $\alpha \geq 1/2$, and a value $\epsilon > 0$

Output: a $(1 - \epsilon)$ -approximate solution

```

1: sort the items by non-increasing value
2:  $\bar{q} \leftarrow \min \{m, \lceil 5(1 - \alpha - \epsilon)/\epsilon \rceil\}$ 
3:  $\mathbf{S} \leftarrow (\emptyset, \emptyset, \emptyset), \text{maxWelf} \leftarrow 0$ 
4: for any allocation with items sale  $\mathbf{X} := (X_0, X_1, X_2)$  restricted to the first  $\bar{q}$  items do
5:    $\mathbf{Y} := (Y_0, Y_1, Y_2) \leftarrow \text{EXTEND}(I, \bar{q}, \mathbf{X})$ 
6:   if  $SW(\mathbf{Y}) > \text{maxWelf}$  then
7:      $\mathbf{S} \leftarrow \mathbf{Y}, \text{maxWelf} \leftarrow SW(\mathbf{Y})$ 
8:   end if
9: end for
10: return  $\mathbf{S}$ 

```

258 is possible, we are able to show that EXTEND returns an EF-IS allocation which, besides
259 the items in X_0 , it also sells items that are worth a total value of at most $5v(\bar{q} + 1)$. If this
260 is not possible, we show that the starting guessed allocation \mathbf{X} cannot be extended to an
261 EF-IS allocation, and so EXTEND returns the basic EF-IS allocation. Upon enumeration
262 of all possible allocations with items sale restricted to the first \bar{q} items, SOLVEBESTEF-IS
263 returns the EF-IS allocation with the largest social welfare.

264 The core of SOLVEBESTEF-IS is the function EXTEND, based on a careful analysis of
265 the various cases that may arise. We provide here an overview of how it works and refer the
266 reader to full version for further details.

267 **Lines 1–6.** The first part of the algorithm checks whether \mathbf{X} is an EF-IS allocation
268 restricted to the first \bar{q} items of I . If this is the case, we invoke COMPLETE to obtain an
269 EF-IS allocation for I . In the negative case, if no items are left to extend \mathbf{X} (i.e., $\bar{q} = m$),

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270 we return the basic EF-IS allocation; otherwise, we conclude that $v_0(X_0) < |v(X_1) - v(X_2)|$
271 and define \bar{X} as the set of the remaining $m - q$ items.

272 **Lines 7–13.** To proceed, line 7 (possibly) swaps X_1 and X_2 so as to have $v(X_1) >$
273 $v(X_2) + v_0(X_0)$. The function now checks whether the items in \bar{X} can be used to obtain an
274 EF-IS allocation out of \mathbf{X} . If $v(X_1) > v(X_2) + v_0(X_0) + v(\bar{X})$, this is not possible and the
275 basic EF-IS allocation is returned (lines 8–10). Otherwise, $v(X_1) \leq v(X_2) + v_0(X_0) + v(\bar{X})$.
276 If this holds at equality, then $(X_0, X_1, X_2 \cup \bar{X})$ is an EF-IS allocation and is returned (lines
277 11–13).

278 **Lines 15–27.** We now have $v(X_1) < v(X_2) + v_0(X_0) + v(\bar{X})$. The first goal here is
279 to check if selling a single item from \bar{X} suffices. Thus, in the while-loop at lines 16–23,
280 the function considers sequentially all items whose removal from \bar{X} invalidates inequality
281 $v(X_1) < v(X_2) + v_0(X_0) + v(\bar{X})$. If, by selling item j , enough money can be raised to cover
282 the difference between $v(X_1)$ and $v(X_2) + v_0(X_0) + v(\bar{X} \setminus \{j\})$, then the EF-IS allocation
283 $(X_0 \cup \{j\}, X_1, X_2 \cup \bar{X} \setminus \{j\})$ is returned (line 18). Otherwise, item j is added to a special
284 set of items S (line 20). If the while-loop terminates without returning a solution, we shall
285 prove that the only way to possibly extend the guessed allocation \mathbf{X} to an EF-IS allocation
286 for I is to assign the items of S to X_2 . Lines 25–27 check whether the new partial allocation
287 after the addition of S to X_2 is EF-IS restricted to the first $q + |S|$ items. If so, COMPLETE
288 is invoked to obtain an EF-IS allocation for I .

289 **Lines 29–31.** If the execution arrives at line 29, it must be either $v(X_1) > v(X_2) + v_0(X_0)$
290 or $v(X_2) > v(X_1) + v_0(X_0)$. In the latter case, we are essentially in the same situation as
291 at the beginning of the function, except for the fact that \mathbf{X} has been extended to the first
292 $q + |S|$ items. Observe that, since we started with $v(X_1) > v(X_2) + v_0(X_0)$ and arrived at a
293 situation in which $v(X_2) > v(X_1) + v_0(X_0)$, there was some progress in between, i.e., $S \neq \emptyset$.
294 Thus, the function sets $q = q + |S|$ and restarts from line 1.

295 **Lines 33–44.** This is the final phase of the algorithm. If we arrive at line 33, we have
296 $v(X_1) > v(X_2) + v_0(X_0)$ and $v(X_1) < v(X_2) + v_0(X_0) + v(\bar{X})$. The while-loop at lines
297 33–35 keeps adding items to the second bundle and stops as soon as an item j is added to
298 X_2 , such that $v(X_1) \leq v(X_2) + v_0(X_0)$. If this condition holds at equality, we can again
299 invoke COMPLETE to obtain an EF-IS allocation for I (lines 36–38). If the execution arrives
300 at line 41, we have $v(X_1) < v(X_2) + v_0(X_0)$ and $v(X_1) \geq v(X_2 \setminus \{j\}) + v_0(X_0)$. At lines
301 41–43, the function computes a set of items Y to be sold guaranteeing that the allocation
302 $(X_0 \cup Y, X_1, X_2 \setminus \{j\})$ is EF-IS. We prove that such a set Y does exist, and by invoking
303 COMPLETE, the algorithm terminates with an EF-IS allocation for I (line 44).

304 The following lemma shows the correctness and complexity of function EXTEND.

305 **► Lemma 8.** *For any input $(I = (m, \mathbf{v}, \mathbf{v}_0), q, X_0, X_1, X_2)$, EXTEND returns an EF-IS*
306 *allocation for I in $O(m)$ time. If there exists an EF-IS allocation for I allocating the first*
307 *q items as in (X_0, X_1, X_2) , then an EF-IS allocation selling items that are worth at most*
308 *$v(X_0) + 5v(q + 1)$ is returned; otherwise, the basic EF-IS allocation is returned.*

■ **Algorithm 3** EXTEND($m, v, v_0, q, X_0, X_1, X_2$)

Input: an ordered instance $I = (m, v, v_0)$ with $\alpha \geq 1/2$, a positive integer $q < m$ and an allocation $\mathbf{X} = (X_0, X_1, X_2)$ restricted to the first q items of I

Output: an EF-IS allocation for I

```

1: if  $v_0(X_0) \geq |v(X_1) - v(X_2)|$  then
2:   return COMPLETE( $I, q, \mathbf{X}$ )
3: else if  $q = m$  then
4:   return ( $[m], \emptyset, \emptyset$ )
5: end if
6:  $\bar{X} \leftarrow [m] \setminus (X_0 \cup X_1 \cup X_2)$ 
7: swap  $X_1$  and  $X_2$  so that  $v(X_1) > v(X_2) + v_0(X_0)$ 
8: if  $v(X_1) > v(X_2) + v_0(X_0) + v(\bar{X})$  then
9:   return ( $[m], \emptyset, \emptyset$ )
10: end if
11: if  $v(X_1) = v(X_2) + v_0(X_0) + v(\bar{X})$  then
12:   return ( $X_0, X_1, X_2 \cup \bar{X}$ )
13: end if
14: % if we reach this point, then  $v(X_1) > v(X_2) + v_0(X_0)$  and  $v(X_1) < v(X_2) + v_0(X_0) + v(\bar{X})$ 

15:  $j \leftarrow q + 1, S \leftarrow \emptyset$ 
16: while  $j \leq m$  &&  $v(j) \geq v(X_2) + v_0(X_0) + v(\bar{X}) - v(X_1)$  do
17:   if  $v_0(j) \geq v(X_1) - v(X_2) - v_0(X_0) - v(\bar{X}) + v(j)$  then
18:     return ( $X_0 \cup \{j\}, X_1, X_2 \cup \bar{X} \setminus \{j\}$ )
19:   else
20:      $S \leftarrow S \cup \{j\}$ 
21:   end if
22:    $j \leftarrow j + 1$ 
23: end while
24:  $X_2 \leftarrow X_2 \cup S, \bar{X} \leftarrow \bar{X} \setminus S$ 
25: if  $v_0(X_0) \geq |v(X_1) - v(X_2)|$  then
26:   return COMPLETE( $I, q + |S|, \mathbf{X}$ )
27: end if
28: % if we reach this point, then either  $v(X_1) > v(X_2) + v_0(X_0)$  or  $v(X_2) > v(X_1) + v_0(X_0)$ 

29: if  $v(X_2) > v(X_1) + v_0(X_0)$  then
30:   set  $q \leftarrow q + |S|$  and goto line 1
31: end if
32: % if we reach this point,  $v(X_1) > v(X_2) + v_0(X_0)$  and, by line 14 and the fact that in
   between items are only moved from  $\bar{X}$  to  $X_2$ ,  $v(X_1) < v(X_2) + v_0(X_0) + v(\bar{X})$ 
33: while ( $v(X_1) > v(X_2) + v_0(X_0)$ ) do
34:    $X_2 \leftarrow X_2 \cup \{j\}, j \leftarrow j + 1$ 
35: end while
36: if ( $v(X_1) = v(X_2) + v_0(X_0)$ ) then
37:   return COMPLETE( $I, j - 1, \mathbf{X}$ )
38: end if
39:  $Y \leftarrow \{j - 1\}, X_2 \leftarrow X_2 \setminus \{j - 1\}$ 
40: % if we reach this point, then  $v(X_1) < v(X_2) + v_0(X_0)$  and  $v(X_1) \geq v(X_2 \setminus \{j\}) + v_0(X_0)$ 

41: while ( $v_0(Y) < v(X_1) - v(X_2) - v_0(X_0)$ ) do
42:    $Y \leftarrow Y \cup \{j\}, j \leftarrow j + 1$ 
43: end while
44: return COMPLETE( $I, j - 1, X_0 \cup Y, X_1, X_2$ )

```

92:10 Achieving Envy-Freeness through Items Sale

309 To prove Lemma 8, we need first some auxiliary results that establish some basic properties
310 of the whole algorithm.

311 ► **Lemma 9.** *Suppose EXTEND reaches line 24, after constructing the set of items S , via*
312 *line 20. If there exists an EF-IS allocation for I allocating the first q items of $[m]$ as in*
313 *(X_0, X_1, X_2) , then it allocates the first $q + |S|$ items of $[m]$ as in $(X_0, X_1, X_2 \cup S)$.*

314 ► **Lemma 10.** *The while-loop at lines 33–35 always terminates; moreover, the while-loop at*
315 *lines 41–43 always terminates returning a set Y such that $v(Y) \leq 3v(q + 1)$.*

316 **Proof of Lemma 8.** First of all, by Lemma 10, we are guaranteed that EXTEND always
317 terminates. Moreover, each returned allocation is always EF-IS either by definition, or by
318 inspection combined with Lemma 7. Regarding the complexity, it is not difficult to see that
319 EXTEND can be executed in $O(m)$ time. In fact, the first 13 lines of the function, apart from
320 basic, constant-time operations, require the computation of quantities such as \bar{X} , $X_2 \cup \bar{X}$,
321 $v(X_0)$, $v(X_1)$, $v(X_2)$, $v(\bar{X})$, taking $O(m)$ time, and the invocation of COMPLETE which, by
322 Lemma 7, needs $O(m)$ time. From line 14 onwards, the function essentially processes all
323 items sequentially and, for each processed item, constant-time operations are performed. The
324 only non-constant-time operation is the invocation to COMPLETE which requires $O(m)$ and
325 is performed only once during any execution of EXTEND.

326 Now, let us bound the value of sold items in any returned solution other than the basic
327 one. The solution returned at line 2 sells items that are worth a total value of at most
328 $v(X_0) + 2v(q + 1)$, due to Lemma 7; the one returned at line 12 sells items that are worth
329 a total value of $v(X_0)$; that returned at line 18, sells items that are worth a total value of
330 $v(X_0) + v(j) \leq v(X_0) + v(q + 1)$; those returned at lines 26 and 37, sell items that are worth
331 in total at most $v(X_0) + 2v(q + 1)$, by Lemma 7; finally, the one returned at line 44 sells
332 items that are worth a total value of at most $v(X_0) + 5v(q + 1)$, by Lemmas 7 and 10.

333 We are left to show that EXTEND returns the basic EF-IS allocation only if no EF-IS
334 allocation extending $\mathbf{X} = (X_0, X_1, X_2)$ exists. EXTEND returns the basic EF-IS allocation
335 only at lines 4 and 9. Observe that, when this happens, there is no chance of getting an
336 EF-IS allocation from \mathbf{X} . This partial allocation, in fact, is either given in input, or obtained
337 from the input after possible repeated additions of the special set S of items to X_2 , at line
338 24. By Lemma 9, these additions do not prevent the extension of the current allocation to
339 an EF-IS one. So, whenever we reach the point that EXTEND has to return the basic EF-IS
340 allocation, it is because it has arrived at an allocation which cannot be extended to become
341 EF-IS and such an allocation has been obtained by performing unavoidable choices only. ◀

342 Putting everything together, we can show our main result.

343 ► **Theorem 11.** *There is a PTAS for BEST-EF-IS with two agents when $\alpha \geq 1/2$.*

Proof. Fix an instance $I = (m, \mathbf{v}, \mathbf{v}_0)$ of BEST-EF-IS. By Lemma 8, we get that, for any
 $0 < \epsilon < 1 - \alpha$, SOLVEBESTEF-IS returns an EF-IS allocation in $O(3^{\bar{q}}m + m \log m)$ time,
with $\bar{q} = O(1/\epsilon)$. To show the approximation guarantee, let \mathbf{O} be the optimal allocation
of the problem, and let (X_0^*, X_1^*, X_2^*) be the partial allocation corresponding to \mathbf{O} and
restricted to the first \bar{q} items. Set $\bar{X}^* := [m] \setminus (X_0^* \cup X_1^* \cup X_2^*)$. When SOLVEBESTEF-IS
calls EXTEND with input I , \bar{q} and (X_0^*, X_1^*, X_2^*) , it receives an EF-IS allocation \mathbf{S} where,
additionally to X_0^* , a set of items $Z \subseteq \bar{X}^*$ is sold and such that $v(Z) \leq 5v(\bar{q} + 1)$, by Lemma
8. We derive $SW(\mathbf{S}) = v(X_1^* \cup X_2^*) + v_0(X_0^*) + v_0(Z) + v(\bar{X}^* \setminus Z)$. On the other hand,
 $SW(\mathbf{O}) \leq v(X_1^* \cup X_2^*) + v_0(X_0^*) + v(Z) + v(\bar{X}^* \setminus Z)$. So, the approximation guarantee

achieved by \mathbf{S} is at least

$$\frac{v(X_1^* \cup X_2^*) + v_0(X_0^*) + v_0(Z) + v(\bar{X}^* \setminus Z)}{v(X_1^* \cup X_2^*) + v_0(X_0^*) + v(Z) + v(\bar{X}^* \setminus Z)} \geq \frac{v(X_1^* \cup X_2^*) + v_0(X_0^*) + \alpha v(Z) + v(\bar{X}^* \setminus Z)}{v(X_1^* \cup X_2^*) + v_0(X_0^*) + v(Z) + v(\bar{X}^* \setminus Z)}.$$

344 This value is minimized when the common terms are as small as possible, while $v(Z)$ is as
 345 large as possible. This is achieved when $v(Z) = 5v(\bar{q} + 1)$, $X_1^* = X_2^* = \bar{X}^* \setminus Z = \emptyset$ and
 346 $X_0^* = [\bar{q}]$, with $v(X_0^*) = \bar{q}v(\bar{q} + 1)$. We derive that the approximation guarantee achieved by
 347 \mathbf{S} is at least $\frac{\bar{q}+5\alpha}{\bar{q}+5} \geq 1 - \epsilon$, for each $\bar{q} \geq \frac{5(1-\alpha-\epsilon)}{\epsilon}$. As SOLVEBESTEF-IS returns the EF-IS
 348 allocation with the highest social welfare, among the ones returned by EXTEND, the claim
 349 follows. \blacktriangleleft

350 \blacktriangleright **Remark 12.** At a first glance, it seems that the value of α plays a major role only within
 351 function COMPLETE. This function can be easily extended to any $\alpha \leq 1/k$, with k being
 352 an integer such that $k > 2$, by selling the next k items, rather than simply the next two
 353 ones. However, the fact that $\alpha \geq 1/2$, is fundamental to prove that the while loop at lines
 354 41–43 of function EXTEND always terminates (see second part of the claim of Lemma 10).
 355 So, in order to extend the PTAS below the threshold $1/2$, additional arguments need to be
 356 elaborated.

357 3.3 An Exact Algorithm for a Small Number of Distinct Values

358 In this subsection, we consider the case in which there is a small number of distinct item values.
 359 In particular, we assume that there are T distinct item values, say $w_1 < w_2 < \dots < w_T$,
 360 and that, for any $s \in [T]$, there are m_s items of value w_s . Obviously, it must hold that
 361 $\sum_{s \in [T]} m_s = m$. We design a dynamic programming algorithm that solves BEST-EF-IS in
 362 polynomial-time when T is a constant.

363 \blacktriangleright **Theorem 13.** *Let T be the number of distinct item values. BEST-EF-IS can be solved in*
 364 *time $O(n(m/T)^{2^T T})$.*

365 4 Extensions to Heterogeneous Agents

366 In this section, we consider a generalization of our model to the case where the items may not
 367 have the same value for all agents. The most natural extension is the one in which each agent
 368 i has her own additive valuation function v_i , so that $v_i(j)$ is the value of agent i for item j
 369 and $\mathbf{v}_i = (v_i(j))_{j \in [m]}$ denotes the vector of all item values for agent i . Under heterogeneous
 370 valuation functions, we need to be more careful about the market value vector \mathbf{v}_0 . As also
 371 done in [22], we assume that for every item j , the market value satisfies $v_0(j) \leq \min_i v_i(j)$.
 372 We view this as a minimal assumption, that should hold so that no agent can have more
 373 value by selling an item rather than by owning it. Furthermore, we assume w.l.o.g. that
 374 the considered allocation problems do not contain any dummy item j with $v_i(j) = 0$ for all
 375 $i \in [n]$.

376 The following result, due to [20] but recast in our framework, shows how to determine in
 377 polynomial time if a given allocation with items sale is EF-IS.

378 \blacktriangleright **Lemma 14** ([20]). *Given an allocation problem $(n, m, (\mathbf{v}_i)_{i \in [n]}, \mathbf{v}_0)$ with heterogeneous*
 379 *valuations, let $\mathbf{X} = (X_0, X_1, \dots, X_n)$ be a an allocation with items sale of $[m]$. One can*
 380 *check in $O(mn + n^3)$ time if \mathbf{X} is EF-IS.*

381 Let $\alpha := \min_{i \in [n], j \in [m]: v_i(j) > 0} \left\{ \frac{v_0(j)}{v_i(j)} \right\} \in [0, 1]$ be the parameter defined similarly as in
 382 Section 3. As before, the basic EF-IS allocation is a feasible solution and trivially constitutes
 383 an optimal one when $\alpha = 1$. Moreover, the hardness results of Section 3.1 continue to hold
 384 under heterogeneous valuations. Therefore the problem is NP-hard, and with 3 agents or more,
 385 there is no approximation factor better than α . Nevertheless, we are still able to provide some
 386 positive results under certain assumptions. In particular, let $\beta := \max_{i \in [n]} \frac{\max_{j \in [m]} v_i(j)}{\min_{j \in [m]: v_i(j) > 0} v_i(j)}$,
 387 denote the maximum ratio between the highest and the lowest (non-zero) valuable item of
 388 any agent. We obtain below a PTAS, if n, β and $1/\alpha$ are bounded by a constant. This can
 389 be seen as generalizing the PTAS of Section 3.2, even beyond the two agent case, but only
 390 with a constant β . The technique that we use however is quite different from the PTAS of
 391 Section 3.2. Furthermore, we also obtain an exact pseudo-polynomial time algorithm for the
 392 case in which the number of agents is constant, without any further assumption.

393 4.1 A PTAS for Few Heterogeneous Agents with $\beta = O(1)$ and $\alpha = \Omega(1)$

394 Within this section we assume that n, β and $1/\alpha$ are bounded by some constant. Moreover,
 395 given two integers k, n , we denote by $[n]_k$ the set $\{k, k + 1, \dots, n\}$, if $k \leq n$ and the empty
 396 set otherwise.

397 In Algorithm 4, we provide the pseudo-code of our PTAS, which we call CUT&SELL. We
 398 provide here a brief overview and intuition of how it works. For a fixed $\epsilon > 0$, we parameterize
 399 the analysis with a constant, but sufficiently large integer q (defined in Algorithm 4), with
 400 $q = O(1/\epsilon)$. If $m \leq q$ the algorithm computes, via a polynomial-time brute-force search, an
 401 EF-IS allocation \mathbf{X} that maximizes the social welfare. Otherwise, when $m > q$, it executes
 402 the following procedures:

403 **Cut Procedure:** This procedure computes a fractional allocation $\mathbf{z} = (z_{i,j})_{i \in [n], j \in [m]}$, with
 404 $z_{i,j}$ denoting the fraction of j assigned to i , that maximizes the social welfare, is envy-free,
 405 and cuts at most $n^2 - n + 1$ items into fractional pieces. To do this in polynomial time,
 406 one can resort to linear programming. In particular, we have that the optimal solution
 407 of the following linear program in variables $(z_{i,j})_{i \in [n], j \in [m]}$ corresponds to a social welfare
 408 maximizing fractional allocation, subject to envy-freeness:

$$\begin{aligned}
 \max \quad & SW(\mathbf{z}) := \sum_{i \in [n], j \in [m]} v_{i,j} z_{i,j} \\
 \text{s.t.} \quad & \sum_{j \in [m]} v_{i,j} z_{i,j} \geq \sum_{j \in [m]} v_{i,j} z_{h,j}, \quad \forall i, h \in [n], i \neq h \\
 & \sum_{i \in [n]} z_{i,j} = 1, \quad \forall j \in [m] \\
 & z_{i,j} \geq 0, \quad \forall i \in [n], j \in [m].
 \end{aligned} \tag{2}$$

410 By Lemma 15 below, we can efficiently find the desired optimal solution of linear program
 411 (2).

412 **► Lemma 15.** *An optimal solution $\mathbf{z} = (z_{i,j})_{i \in [n], j \in [m]}$ of linear program (2) that satisfies*
 413 *$|\{j \in [m] : \exists i \in [n], 0 < z_{i,j} < 1\}| \leq n^2 - n + 1$ can be computed in polynomial time.*

414 **Sell Procedure:** Let X_0 be the set of items that are assigned fractionally in \mathbf{z} . We first
 415 sell X_0 and derive the (integral) allocation with items sale $\mathbf{X} = (X_0, X_1, \dots, X_n)$, where
 416 each j in X_i is assigned integrally to i in \mathbf{z} . Now, given the allocation $\mathbf{X}' = (X_1, \dots, X_n)$

restricted to the unsold items of \mathbf{X} , the *envy-graph* of \mathbf{X}' is a graph $G_{\mathbf{X}'} = (V, E_{\mathbf{X}'})$ having $V = [n]$, and containing an edge $(i, h) \in [n]^2$ if and only if agent i is envious of h (that is, $v_i(X_i) < v_i(X_h)$). The algorithm permutes the bundles in such a way that the resulting envy-graph of the allocation restricted to unsold items is acyclic. This step is implemented by a sub-routine called `ENVYCYCLEELIMINATION` introduced in [24] (we refer the reader to such work for a detailed description of the sub-routine). Then, the algorithm alternates among the following two steps, until the obtained allocation \mathbf{X} is EF-IS: (i) It picks a sink $i \in [n]$ in the resulting (acyclic) envy-graph, i.e., an agent that is not currently envious of anyone. Then, the algorithm removes and sells an arbitrary good j from the bundle of i (i.e., j is added to X_0). (ii) It applies again `ENVYCYCLEELIMINATION`, so that the resulting envy-graph of the allocation restricted to unsold items becomes acyclic.

We shall prove that at the end, the algorithm terminates with an EF-IS allocation and with the desired approximation.

► **Theorem 16.** *There is a PTAS for BEST-EF-IS with n heterogeneous agents, when n, β and $1/\alpha$ are all $O(1)$.*

Proof Sketch. We first argue about the complexity of the algorithm. If $m \leq q$, `CUT&SELL` enumerates $(n+1)^m \leq (n+1)^q = O(1)$ distinct allocations with item sales; for each of them, it verifies in polynomial time if it is EF-IS (by Lemma 14), and finally returns the one of maximum welfare. The case $m > q$ is polynomial as it requires to find an optimal fractional allocation via linear programming and applies at most $O(m)$ times the envy-cycle-elimination procedure.

The technically more involved part is to show that `CUT&SELL` guarantees a $(1 - \epsilon)$ -approximation. If $m \leq q$, an optimal solution is returned by enumeration. If $m > q$, `CUT&SELL` first computes an optimal fractional envy-free-solution \mathbf{z} , whose social welfare $SW(\mathbf{z})$ is used as an optimality benchmark for BEST-EF-IS. Then, by using the hypothesis that $n, 1/\alpha, \beta$ are bounded by a constant, we show that the items with positive value added to X_0 during the `SELL` procedure, before reaching an EF-IS allocation, have low value compared to $SW(\mathbf{z})$. This fact is used to show that the allocation returned by the algorithm is a $(1 - \epsilon)$ -approximation. ◀

4.2 A Pseudo-Polynomial Time Algorithm for Few Heterogeneous Agents

Our final result is that for a constant number of agents, BEST-EF-IS admits a pseudo-polynomial time algorithm. Considering the hardness result provided in Theorem 6, a limitation on the number of agents is necessary to obtain a pseudo-polynomial time algorithm, unless $P = NP$.

► **Theorem 17.** *BEST-EF-IS can be solved in $O(mn^2V^{n^2})$ time for heterogeneous valuations and in $O(mnV^n)$ time for identical ones, where $V = \max_{i \in [n]} \{v_i([m])\}$ denotes the maximum value for the entire set of items.*

5 Conclusions and Future work

Our work explores from an algorithmic perspective the model of fair division of indivisible items initiated in [22], and provides an almost complete picture on its status. This model considers the possibility of selling items in order to compensate envious agents in a proposed allocation.

Algorithm 4 CUT&SELL($m, \mathbf{v}, \mathbf{v}_0$)

Input: $\epsilon > 0$ and an instance $I = (m, \mathbf{v}, \mathbf{v}_0)$ with heterogeneous agents, $\alpha \in (0, 1)$, $\beta \geq 1$ and $v_i(j) > 0$ for any $i \in [n]$, $j \in [m]$

Output: an EF-IS allocation for I

```

1:  $q \leftarrow \lceil n(n^2 + 1)\beta(\beta - \alpha)/(\epsilon\alpha^2) \rceil$ 
2: if  $m \leq q$  then
3:   Find the best EF-IS allocation by enumeration
4: else
5:   % CUT Procedure
6:    $\mathbf{z} = (z_{i,j})_{i \in [n], j \in [m]} \leftarrow$  an optimal envy-free fractional allocation, in which at most
    $n^2 - n + 1$  items are cut into two or more fractional pieces.
7:   % SELL Procedure
8:    $X_0 \leftarrow \{j \in [m] : \exists i, \text{ s.t. } z_{i,j} \in (0, 1), i \in [n]\}$ 
9:   for  $i = 1, \dots, n$  do
10:     $X_i \leftarrow \{j \in [m] : z_{i,j} = 1\}$ 
11:   end for
12:    $\mathbf{X} = (X_0, \dots, X_n)$ ,  $\mathbf{X}' \leftarrow (X_1, \dots, X_n)$ 
13:    $\mathbf{X}' \leftarrow \text{ENVYCYCLEELIMINATION}(\mathbf{X}')$ 
14:   while  $\mathbf{X}$  is not EF-IS do
15:     Let  $i \in [n]$  such that  $v_i(X_i) \geq v_i(X_h)$  for each  $h \in [n]$  (break ties arbitrarily)
16:     Let  $j \in X_i$  (break ties arbitrarily)
17:      $X_i \leftarrow X_i \setminus \{j\}$ ,  $X_0 \leftarrow X_0 \cup \{j\}$ 
18:      $\mathbf{X}' \leftarrow (X_1, \dots, X_n)$ 
19:      $\mathbf{X}' \leftarrow \text{ENVYCYCLEELIMINATION}(\mathbf{X}')$ 
20:   end while
21: end if
22: return  $\mathbf{X}$ 

```

459 Despite the large amount of research work devoted in the last years to the study of
460 relaxed notions of envy-freeness, the approach of items sale has remained largely unexplored.
461 This may look strange since, although relaxed notions of envy-freeness such as EFX and
462 EF1 provide theoretically interesting and elegant solutions to the non-existence of envy-free
463 allocations, from a practical point of view there are many cases in which these solutions
464 are highly unfair (think, for instance, of the famous basic case of a high-valued item and
465 two agents). A possible reason for this under-consideration might come from the intrinsic
466 difficulty of the problem, as witnessed by the strong computational barriers we proved in
467 Subsection 3.1. However, we have also shown that, under some (in some cases even mild)
468 assumptions, interesting positive results are possible.

469 An interesting open question that arises is whether we can extend the existence of a
470 PTAS for two agents, in the case of $\alpha \in (0, 1/2)$ and identical valuations, and also in the
471 case of arbitrary α and heterogeneous valuations (without further assumptions on other
472 parameters). Furthermore, it would be nice to study the effects of items sale for other variants
473 of fair allocation problems, such as for other notions of fairness (e.g., proportionality, EFX
474 or maximin shares) or for more general valuations beyond additivity, or for problems with
475 additional constraints (e.g., under connectivity constraints [10, 11, 26]). Finally, it would be
476 interesting to study the case of strategic agents, as in [4], who may misreport their valuations
477 to increase their utility.

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